Study of a nonconservative product for a thick spray model

Victor Fournet - LJLL, CEA
fournet@ljll.math.upmc.fr

Introduction

We are interested in thick spray models [1], which are similar to system (4). In the usual formulation of the model, the Euler part admits a conservative formulation, but the Vlasov part does not. This fact brings the question of the definition of weak solution for such systems. To study the nonconservative product in the Vlasov equation, we consider a toy model :

$$\partial_t f + v\partial_x f - \partial_x p_\varepsilon \partial_v f = 0, \quad \text{where the pressure is a travelling wave: } p_\varepsilon(x,t) = p\left(\frac{x-\sigma t}{\varepsilon}\right), \quad p(\pm\infty) = p^\pm, \quad p' < 0. \tag{1}$$

Approach a la Dal Maso-Le Floch-Murat

A first approach is to study the nonconservative product with the DLM theory [2]. In this theory, the nonconservative product of the form $g\partial_x u$ is defined as a measure, denoted $[g\partial_x u]_\phi$ which depends on a family of Lipschitz path ϕ. Once a path ϕ has been chosen, one writes $\forall \psi \in \mathcal{D}(\mathbf{R}^2)$

$$\int_{\mathbf{R}^2} \partial_x p_\varepsilon \partial_v f \psi \, dxdv = -\int_{\mathbf{R}^2} \partial_x p_\varepsilon f \partial_v \psi \, dxdv \xrightarrow[\varepsilon \to 0]{} -\int_{\mathbf{R}^3} [\partial_x p f]_\phi \, \partial_v \psi \, dxdv.$$

Definition 1 *Let ϕ be a Lipschitz continuous path. We say that f is a DLM-weak solution of the Vlasov equation (1) if, for every $\psi \in \mathcal{D}(\mathbf{R}^3)$,*

$$\int_{\mathbf{R}^3} (\partial_t \psi(t,x,v) + v\partial_x \psi(t,x,v)) f(t,x,v) \, dxdvdt - \int_{\mathbf{R}^3} [\partial_x p f]_\phi \, \partial_v \psi(t,x,v) \, dxdvdt = 0.$$

We study the relevance of this definition, regarding the fact that if it contains simple physical solution of the form of a travelling wave.

Proposition 1 *Let ϕ be a Lipschitz continuous path and f of the form $f(x,v,t) = g(x-\sigma t, v-\sigma) = \mathbf{1}_{[x-\sigma t,+\infty[\times[v_{1,+}-\sigma, v_{2,+}-\sigma]}(x,v) + \mathbf{1}_{[x-\sigma t,+\infty[\times[v_{1,-}-\sigma, v_{2,-}-\sigma]}(x,v)$.*
There exist $\psi \in \mathcal{D}(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3} (\partial_t \psi(t,x,v) + v\partial_x \psi(t,x,v)) \, f(t,x,v) \, dxdvdt \tag{2}$$

$$- \int_{\mathbf{R}^3} [\partial_x p f]_\phi \, \partial_v \psi(t,x,v) \, dxdvdt \neq 0. \tag{3}$$

That is, for any ϕ, there exists a travelling wave for which the criterion does not hold at the limit.

An averaged model

All those results motivate us to change the model, so that we can define usual weak solutions. We propose the following system [1] :

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla}p = m_\star \boldsymbol{\nabla}p \int \langle f\rangle dv + D_\star \int (\boldsymbol{v}-\boldsymbol{u}) f dv \\ \partial_t(\alpha\varrho e) + \boldsymbol{\nabla}\cdot(\alpha\varrho e\boldsymbol{u}) + p(\partial_t\alpha + \boldsymbol{\nabla}\cdot(\alpha\boldsymbol{u})) = D_\star \int |\boldsymbol{v}-\boldsymbol{u}|^2 f dv \\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_v\cdot(\Gamma f) = 0 \\ \alpha = 1 - m_\star \int \langle f\rangle dv \\ m_\star \Gamma = -m_\star \langle \boldsymbol{\nabla}p\rangle - D_\star(\boldsymbol{v}-\boldsymbol{u}) \end{cases} \tag{4}$$

with $\langle\cdot\rangle$ defined by $\langle f\rangle(x) = \int_{\mathbf{R}} w(x-y)f(y)dy$.
One has the entropy law

$$\partial_t(\alpha\varrho S) + \boldsymbol{\nabla}\cdot(\alpha\varrho S\boldsymbol{u}) = \frac{D_\star}{T}\int |\boldsymbol{v}-\boldsymbol{u}|^2 f dv \geq 0.$$

Proposition 3 *The system is conservative in total mass, total momentum and total energy.*

Proposition 4 ([3]) *Under some boundedness assumption, and assuming that*

$$\boldsymbol{u} \in W^{1,\infty}, \quad \frac{\int f\boldsymbol{v}dv}{\int \langle f\rangle dv} \in L^\infty,$$

then the volume fraction stays positive for smooth solution.

Theorem 1 ([1]) *The system is locally well posed in Sobolev spaces H^s for small initial data on the particle distribution.*

Limit with the Laplace transform

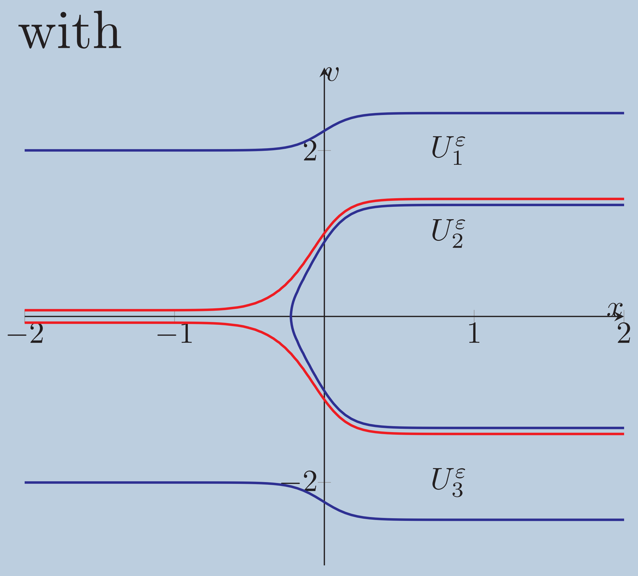
Another approach is the direct study of the regularized equation through a Laplace transform :

$$\mu\hat{f}(x,v,\mu) - f_0(x,v) + v\partial_x\hat{f}(x,v,\mu) - \frac{1}{\varepsilon}p'\left(\frac{x}{\varepsilon}\right)\partial_v\hat{f}(x,v,\mu) = 0.$$

The characteristic curves of $v\partial_x - \frac{1}{\varepsilon}p'\left(\frac{\cdot}{\varepsilon}\right)\partial_v$ split the domain into three regions, U_1^ε, U_2^ε and U_3^ε. We introduce the change of variables $(x,v) \to (S_\varepsilon(x,v), H_\varepsilon(x,v))$ with

$$S_\varepsilon(x,v) = \mathrm{sgn}(v)\int_{x_0(x,v)}^{x} \frac{dy}{\sqrt{2H_\varepsilon(x,v) - 2p_\varepsilon(y)}},$$

$$H_\varepsilon(x,v) = \frac{v^2}{2} + p\left(\frac{x}{\varepsilon}\right).$$



In theses variables we have the following

Lemma 1 *For all (x,v) with $v \neq 0$, we have*

$$v\partial_x S_\varepsilon(x,v) - \frac{1}{\varepsilon}p\left(\frac{x}{\varepsilon}\right)\partial_v S_\varepsilon(x,v) = 1, \quad v\partial_x - \frac{1}{\varepsilon}p\left(\frac{\cdot}{\varepsilon}\right)\partial_v = \partial_S, \quad \partial_S(e^{\mu S}\hat{f}(S,H)) = e^{\mu S}f_0(S,H).$$

In the end, we obtain a representation formula for f using the inverse Laplace transform.
We look for jump relation for f across the shock.

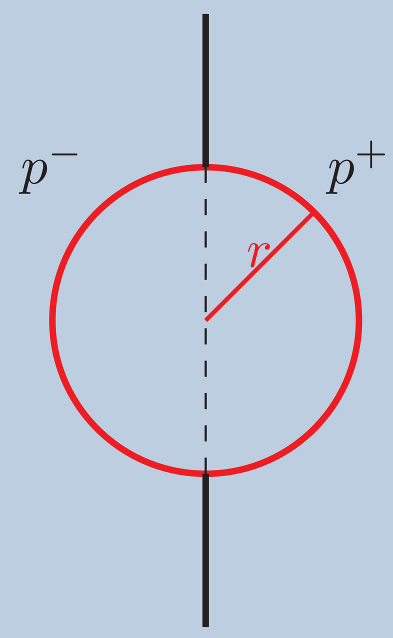
Proposition 2 *In U_1, we have $f(0+,v,t) = f_0(-t\sqrt{v^2+2p^+-2p^-}, \sqrt{v^2+2p^+-2p^-})$ and $f(0-,v,t) = f_0(-tv,v)$.*
In U_2, we have $f(0+,v,t) = f_0(-t|v|,|v|)$, $f(0-,v,t) = 0$

So far we do not know how to write a simple jump relation for f which is independent of f_0.

The convolution kernel

An example of appropriate an convolution kernel w is given by the formula :

$$w(x) = \mathbf{1}_{\mathbf{S}^3}(x).$$



References

[1] Victor Fournet, Christophe Buet, and Bruno Després. Local-in-time existence of strong solutions to an averaged thick sprays model. working paper or preprint, December 2022.

[2] Gianni Dal Maso, Patrick Le Floch, and François Murat. Definition and weak stability of nonconservative products. *Journal de Mathématiques Pures et Appliquées*, 74:483–548, 1995.

[3] C Buet, B Després, and L Desvillettes. Linear stability of thick sprays equations. working paper or preprint, December 2021.